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GYROKINETIC MODELS FOR EDGE PLASMAS

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REQUIREMENTS

- **Conservative**
  - **particles**
  - **energy**
  - **momentum (in the absence of external forces; if translational, rotational, helical symmetry present)**

- **Electromagnetic (low-β; $\delta B_\perp$ only)**

- **Straightforwardly implementable**
  - **no additional kinetic equations, e.g., for S_1, etc.**
  - **no time derivatives on RHS**
  - **explicit pseudo-implementation statements for each term**

SOME REFERENCES

Dubin 1983: D.H.E. Dubin, J.A. Krommes, C. Oberman, W.W. Lee, Phys. Fluids **26**, 3524 (1983) – consistent conservative nonlinear gyrokinetic equations

Hahm 1988a: T.S. Hahm, W. W. Lee, A.J. Brizard Phys. Fluids **31**, 1940 (1988) – electromagnetic

Hahm 1988: T.S. Hahm, Phys. Fluids **31**, 2670 (1988) – toroidal

Dimits 1992: A.M. Dimits, L.L. LoDestro, D.H.E. Dubin, Phys. Fluids B4 274 (1992) – extended ordering (electromagnetic, slab)

Sugama 2000: H. Sugama, Phys. Plasmas **7**, 466 (2000); Brizard 2000: A.J. Brizard, Phys. Plasmas **7**, 4816 (2000) - variational formulations for easy derivation of conservation laws

Brizard 2007: Rev. Mod. Phys **79**, 421 (2007) - review

Qin 2007: H. Qin, R. H. Cohen, W. M. Nevins, and X. Q. Xu, Phys. Plasmas **14**, 056110 (2007) – “small tokamak ordering” – partially edge relevant

Brizard 2004: A.J. Brizard, Phys. Plasmas **11**, 4429 (2004) – bilinear (large-δf) gyrokinetic Coulomb collision operator

CONSEQUENCES OF ENERGY CONSERVATION

For a gyrokinetic Vlasov-Poisson or Vlasov-Maxwell system in which the generalized polarization term in the Poisson equation (and Ampere's Law) is consistent with the actual (gyrocenter) distribution *F*, the Hamiltonian and phase-space Lagrangian (Poincare-Cartan 1 form) must be evaluated to second order in the gyrokinetic smallness parameter ε_{GK} in order for the entire system to satisfy energy conservation. This has been noted by Dubin and Hahm.

If the distribution function in the gyrokinetic Poisson operator (and Ampere's Law) (generalized polarization term) is fixed or approximated in such a way that the operator does not depend on the evolving (gyrocenter) distribution function, (as in the case of core-delta-f models) then energy conservation can be satisfied with particle equations of motion that keep only terms up to first-order in ε_{GK}.

Electromagnetic Gyrokinetic System of Equations for the Edge: Canonical representation

References: A.J. Brizard: Phys. Plasmas **7**, 4816 (2000 - variational formulation & conservation laws); Rev. Mod. Phys **79**, 421 (2007 – review)

Gyrokinetic Vlasov equation in conservation form:

$$\frac{\partial}{\partial t}\left[B_{\|}^{*}F_i\right]+\boldsymbol{\nabla}_R\cdot\left[B_{\|}^{*}F_i\dot{\boldsymbol{R}}\right]+\frac{\partial}{\partial\overline{p}_{\|}}\left[B_{\|}^{*}F_i\dot{\overline{p}}_{\|}\right]=C\,. \qquad (1.1)$$

Equations of motion:

$$\dot{\boldsymbol{R}}=\frac{\boldsymbol{b_0}}{B_{\|}^{*}}\times\boldsymbol{\nabla}_R H+\frac{\boldsymbol{B}^{*}}{B_{\|}^{*}}\frac{\partial H}{\partial\overline{p}_{\|}};\quad \dot{\overline{p}}_{\|}=-\frac{\boldsymbol{B}^{*}}{B_{\|}^{*}}\boldsymbol{\nabla}_R H,$$

$$\boldsymbol{B}^{*}=\boldsymbol{B_0}+\overline{p}_{\|}\boldsymbol{\nabla}\times\boldsymbol{b_0};\quad B_{\|}^{*}\equiv\boldsymbol{B}^{*}.\boldsymbol{b_0}$$

$$H=\frac{\overline{p}_{\|}^{2}}{2}+\mu B_0+\left\langle\psi_1\right\rangle-\frac{1}{2\Omega}\left[\frac{\partial}{\partial\mu}\left\langle(\tilde{\phi}_1)^2\right\rangle+\frac{1}{\Omega^{*}}\left\langle\boldsymbol{\nabla}_{R\perp}\tilde{\Phi}_1\cdot\boldsymbol{b_0}\times\boldsymbol{\nabla}_{R\perp}\tilde{\phi}_1\right\rangle\right]+\frac{1}{2}\left\langle A_{1\|}^{2}\right\rangle, \qquad (1.2)$$

$$\psi_1=\phi_1-\overline{p}_{\|}A_{1\|},$$

Note: The first two second-order (quadratic in perturbation size) terms in H_2 have ϕ_1 not ψ_1. The $\left\langle A_{1\|}^2 \right\rangle$ term is needed for positivity of terms in energy.

The GK Poisson's equation has only electrostatic terms:

$$0 \approx -\frac{1}{4\pi e}\nabla^2\phi(x) = \int d\Lambda\,\delta\left(\boldsymbol{R}+\boldsymbol{\rho}-\boldsymbol{x}\right)\left[F_i + \tilde{\phi}_1\frac{\partial F_i}{\partial \mu} + \boldsymbol{\nabla}_{R\perp}\tilde{\Phi}_1\cdot\boldsymbol{b_0}\times\boldsymbol{\nabla}_{R\perp}F_i\right] - n_e$$

The GK Ampere's Law has only the identity part of T_{gy}

$$\left\{\nabla^2 - \left[\delta\omega_{pe}^2(\boldsymbol{x}) + \delta\omega_{pi}^2(\boldsymbol{x})\right]\Big/c^2\right\}A_\| = \left(4\pi e/m_e\right)\left[\int d\Lambda F_e p_\| - \left(Zm_e/m_i\right)\int d\Lambda\,\delta\left(\boldsymbol{R}+\boldsymbol{\rho}-\boldsymbol{x}\right)F_i p_\|\right]$$

The above equations, and energy conservation, all follow from a single reduced variational principle and Noether's theorem (following Brizard):

$$0 = \delta\mathcal{A}, \text{ wrt. } \quad \delta S_\alpha(\boldsymbol{x},t), \quad \delta\phi_1(\boldsymbol{x},t), \quad \delta A_{1\|}(\boldsymbol{x},t),$$

where

$$\mathcal{A} = -\int d^8\mathcal{Z}\,\mathcal{D}_i(\mathcal{Z})\,\mathcal{F}_i(\mathcal{Z})\mathcal{H}_i(\mathcal{Z}) - \int d^8\mathcal{Z}\,\mathcal{D}_e(\mathcal{Z})\,\mathcal{F}_e(\mathcal{Z})\mathcal{H}_e(\mathcal{Z}) - \int \frac{d^4x}{4\pi}\left|\delta B\right|^2,$$

$$\delta\mathcal{F}_\alpha(\mathcal{Z}) = \left\{F_\alpha, \delta S_\alpha\right\},$$

$$\mathcal{Z} = (\boldsymbol{Z},t,w), \quad \mathcal{H}(\mathcal{Z}) = H(\boldsymbol{Z},t) - w, \quad \mathcal{F}(\mathcal{Z}) = F(\boldsymbol{Z},t)\delta\left(\mathcal{H}(\mathcal{Z})\right),$$

$$\mathcal{D}_\alpha(\mathcal{Z}) = \frac{\partial\mathcal{Z}}{\partial(\boldsymbol{x},\boldsymbol{v},t,w)} = \frac{B_\|^*}{m_\alpha} \text{ - Jacobian}$$

Conserved energy:

$$\mathcal{E}_{gy} = \int d^3p\; D(Z)F_i(H_i - q\left\langle T_{gy}^{-1}\Phi_{gc}\right\rangle) - \int d^3p\; D(Z)F_e(H_e + e\phi) + \frac{1}{8\pi}\left|\delta B\right|^2$$

New Electromagnetic Gyrokinetic System of Equations for MFE Edge Plasmas: Symplectic Representation.

References: A.J. Brizard: Rev. Mod. Phys **79**, 421 (2007 – review); Sugama 2000: H. Sugama, Phys. Plasmas **7**, 466 (2000).

Gyrokinetic Vlasov equation in conservation form:

$$\frac{\partial}{\partial t}\left[B_{\|}^{*}F_i\right]+\boldsymbol{\nabla}_R\cdot\left[B_{\|}^{*}F_i\dot{\boldsymbol{R}}\right]+\frac{\partial}{\partial\overline{p}_{\|}}\left[B_{\|}^{*}F_i\dot{\overline{p}}_{\|}\right]=C\ . \quad (1.3)$$

Equations of motion:

$$\dot{\boldsymbol{R}}=\frac{\boldsymbol{b_0}}{B_{\|}^{*}}\times\boldsymbol{\nabla}_R H+\frac{\boldsymbol{B}^{*}}{B_{\|}^{*}}\frac{\partial H}{\partial\overline{p}_{\|}};\quad \dot{\overline{p}}_{\|}=-\frac{\boldsymbol{B}^{*}}{B_{\|}^{*}}\boldsymbol{\nabla}_R H-\frac{\left\langle\partial A_{1\|}\right\rangle}{\partial t},$$

$$\boldsymbol{B}^{*}=\boldsymbol{B}+\overline{p}_{\|}\boldsymbol{\nabla}\times\boldsymbol{b_0}=\boldsymbol{B_0}+\boldsymbol{B}_{1\perp}+\overline{p}_{\|}\boldsymbol{\nabla}\times\boldsymbol{b_0},$$

$$\boldsymbol{B}_{1\perp}=\boldsymbol{\nabla}\times\left(\left\langle A_{1\|}\right\rangle\boldsymbol{b_0}\right),$$

$$H = \frac{\overline{p}_{\|}^{2}}{2} + \mu B_0 + \langle \phi_1 \rangle - \frac{1}{2\Omega}\left[\frac{\partial}{\partial \mu}\left\langle (\tilde{\phi}_1)^2 \right\rangle + \frac{1}{\Omega^*}\left\langle \boldsymbol{\nabla}_{R\perp}\tilde{\Phi}_1 \cdot \boldsymbol{b_0} \times \boldsymbol{\nabla}_{R\perp}\tilde{\phi}_1 \right\rangle \right] + \frac{1}{2}\left\langle A_{1\|}^{2} \right\rangle, \quad (1.4)$$

Note: The first two second-order (quadratic in perturbation size) terms in H_2 have ϕ_1 not ψ_1. The $\left\langle A_{1\|}^{2} \right\rangle$ term is needed for positivity of terms in energy.

The GK Poisson's equation again has only electrostatic terms:

$$0 \approx -\frac{1}{4\pi e}\nabla^2\phi(x) = \int d\Lambda\, \delta\left(\boldsymbol{R} + \boldsymbol{\rho} - \boldsymbol{x}\right)\left[F_i + \tilde{\phi}_1 \frac{\partial F_i}{\partial \mu} + \boldsymbol{\nabla}_{R\perp}\tilde{\Phi}_1 \cdot \boldsymbol{b_0} \times \boldsymbol{\nabla}_{R\perp} F_i \right] - n_e$$

The GK Ampere's Law has only the identity part of T_{gy}

$$\left(\nabla^2/c^2\right) A_{\|} = \left(4\pi e/m_e\right)\left[\int dV F_e p_{\|} - \left(Z m_e/m_i\right) \int d\Lambda\, \delta\left(\boldsymbol{R} + \boldsymbol{\rho} - \boldsymbol{x}\right) F_i p_{\|} \right]$$

Note that the key difference between the canonical and symplectic/kinetic representations for the electron motion is that in the radial displacement due to magnetic flutter and parallel streaming lies in the $\boldsymbol{b_0} \times \boldsymbol{\nabla}_R H / B_{\parallel}^*$ in the canonical representation, while in the symplectic representation, it lies in the $\left(\boldsymbol{B}^* / B_{\parallel}^*\right)\left(\partial H / \partial p_{\parallel}\right)$ term. Note that we are using a low-β ordering in which only the component of the perturbed magnetic field perpendicular to the equilibrium field is kept.

Progress on a Gyrokinetic Coulomb Collision Operator

A.M. Dimits and the ESL Team

Gyrokinetic Coulomb Collision operators – some background.
X. Q. Xu and M. N. Rosenbluth, Phys. Fluids B **3**, 627 (1991) – Linear, Fourier $\perp$ space, statistical implementation of conservation laws
A. M. Dimits and B. I. Cohen, Phys. Rev. E **49**, 709 (1994). – Linear, configuration space, more accurate scheme for conservation laws.
Z. Lin – extended Dimits-Cohen operator to accurately preserve drifting Maxwellian
A.J. Brizard: Phys. Plasmas **11**, 4429 (2004) formulated bilinear (allows for field-particle distribution to depart from equilibrium) gyrokinetic Coulomb collision operator.

Brizard's paper lays a systematic basis for a bilinear gyrokinetic Coulomb collision operator. However, we note that for gyrokinetic turbulence and neoclassical simulation applications, it is important to retain the first-order (in the gyrokinetic smallness parameter) terms in the pull-back operator (e.g., in Brizard's Eq. [7]. That is, there are leading order corrections to Brizard's Eqs.[23] and [47].) These terms were noted and retained, e.g., by Xu and Rosenbluth and by Dimits and Cohen. This is because the lowest-order part of equilibrium distribution functions annihilates the collision operator in these applications.

IMPLEMENTATION CONSIDERATIONS

Computation of equations of motion (phase-space velocities)

$$
\begin{aligned}
\dot{\boldsymbol{R}} &= \frac{\boldsymbol{b_0}}{B_{\|}^{*}} \times \boldsymbol{\nabla}_R H + \frac{\boldsymbol{B}^{*}}{B_{\|}^{*}} \frac{\partial H}{\partial \overline{p}_{\|}}, \\
\boldsymbol{b_0} &\equiv \boldsymbol{B_0} / B_0, \\
B_{\|}^{*} &\equiv \boldsymbol{B}^{*} . \boldsymbol{b_0}, \\
H &= H_0 + H_1 + H_2, \qquad (1.5) \\
H_0 &= \frac{\overline{p}_{\|}^{2}}{2} + \mu B_0,
\end{aligned}
$$

CANONICAL REPRESENTATION:

$$\boldsymbol{B}^* = \boldsymbol{B_0} + \overline{p}_{\|}\boldsymbol{\nabla}\times\boldsymbol{b_0},$$

$$B_{\|}^* \equiv B_0 + \overline{p}_{\|}\boldsymbol{b_0}\cdot\boldsymbol{\nabla}\times\boldsymbol{b_0},$$

$$\dot{\overline{p}}_{\|} = -\frac{\boldsymbol{B}^*}{B_{\|}^*}\boldsymbol{\nabla}_R H,$$

$$H_1 = \langle\psi_1\rangle,$$

$$H_2 = -\frac{1}{2\Omega}\left[\frac{\partial}{\partial\mu}\left\langle(\tilde{\psi}_1)^2\right\rangle + \frac{1}{\Omega^*}\left\langle\boldsymbol{\nabla}_{R\perp}\tilde{\Psi}_1\cdot\boldsymbol{b_0}\times\boldsymbol{\nabla}_{R\perp}\tilde{\psi}_1\right\rangle\right] + \frac{1}{2}\left\langle A_{1\|}^2\right\rangle,$$

$$\psi_1 = \phi_1 - \overline{p}_{\|}A_{1\|},$$

$$\tilde{\Psi} = \frac{1}{\Omega}\int^{\theta} d\theta\,\tilde{\psi},$$

$$\tilde{\psi} = \psi - \langle\psi\rangle,$$

$$\langle\psi\rangle = \frac{1}{2\pi}\oint d\theta\,\psi,$$

SYMPLECTIC REPRESENTATION:

$$\boldsymbol{B}^* = \boldsymbol{B} + \overline{p}_{\|}\boldsymbol{\nabla}\times\boldsymbol{b_0} = \boldsymbol{B_0} + \boldsymbol{B_{1\perp}} + \overline{p}_{\|}\boldsymbol{\nabla}\times\boldsymbol{b_0},$$

$$B_{\|}^* \equiv \boldsymbol{B}^*.\boldsymbol{b_0} = B_0 + \overline{p}_{\|}\boldsymbol{b_0}\cdot\boldsymbol{\nabla}\times\boldsymbol{b_0},$$

$$\boldsymbol{B_{1\perp}} = \boldsymbol{\nabla}\times\left(\left\langle A_{1\|}\right\rangle\boldsymbol{b_0}\right)$$

$$\dot{\overline{p}}_{\|} = -\frac{\boldsymbol{B}^*}{B_{\|}^*}\boldsymbol{\nabla}_R H - \frac{\left\langle\partial A_{1\|}\right\rangle}{\partial t},$$

$$H_1 = \left\langle\psi_1\right\rangle = \left\langle\phi_1\right\rangle,$$

$$H_2 = -\frac{1}{2\Omega}\left[\frac{\partial}{\partial\mu}\left\langle(\tilde{\psi}_1)^2\right\rangle + \frac{1}{\Omega^*}\left\langle\boldsymbol{\nabla}_{R\perp}\tilde{\Psi}_1\cdot\boldsymbol{b_0}\times\boldsymbol{\nabla}_{R\perp}\tilde{\psi}_1\right\rangle\right], \quad (1.6)$$

$$\overline{p}_{\|} = p_{\|} - \left(A_{1\|} - \left\langle A_{1\|}\right\rangle\right),$$

$$\psi_1 = \phi_1 - \overline{p}_{\|}\left(A_{1\|} - \left\langle A_{1\|}\right\rangle\right).$$

First-order term - given ψ, want $\langle\psi\rangle(\boldsymbol{R},\mu,t)$ either at each mesh node or cell center, or at each gyrocenter (in the case of a PIC code), as well as derivatives needed for the equations of motion. Can calculate directly using sampling of points on the gyroorbit or in Fourier space. Save for use on the 4D $(\boldsymbol{R},\mu)$ mesh. For a PIC code, can do this particle by particle.

Second-order terms:

$\partial\left\langle(\tilde{\psi}_1)^2\right\rangle\big/\partial\mu$: use

$$\frac{\partial\left\langle(\tilde{\psi}_1)^2\right\rangle}{\partial\mu}=\frac{\partial\left\langle\psi_1^2\right\rangle}{\partial\mu}-2\left\langle\psi_1\right\rangle\frac{\partial\left\langle\psi_1\right\rangle}{\partial\mu} \qquad (1.7)$$

and

$$\frac{\partial\langle\psi\rangle}{\partial\mu}=\frac{\partial\rho}{\partial\mu}\frac{1}{2\pi}\oint d\theta\,\hat{\rho}(\theta)\cdot\nabla\,\psi\left(R+\rho\hat{\rho}(\theta)\right). \qquad (1.8)$$

This can be calculated directly by sampling the components of $\nabla\psi$ around an instantaneous gyro orbit or by using a Fourier decomposition and Bessel functions. Again, at any given *t*, this is a function of the 4D $(\boldsymbol{R},\mu)$ phase space.

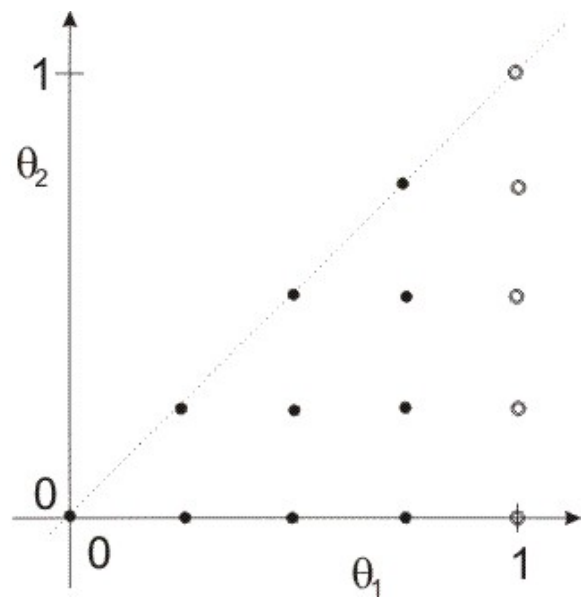
Second-order terms, contd.:

$$\left\langle \nabla_{R\perp}\tilde{\Phi}\cdot \boldsymbol{b_0}\times\nabla_{R\perp}\tilde{\phi}\right\rangle = \boldsymbol{b_0}\cdot\oint d\theta_1\int_0^{\theta_1} d\theta_2\,\nabla_{R\perp}\tilde{\phi}\left(\vec{R}+\rho\hat{\vec{\rho}}(\theta_1)\right)\times\nabla_{R\perp}\tilde{\phi}\left(\vec{R}+\rho\hat{\vec{\rho}}(\theta_2)\right)$$

$$= \boldsymbol{b_0}\cdot\oint d\theta_1\int_0^{\theta_1} d\theta_2\,\nabla_{R\perp}\phi\left(\vec{R}+\rho\hat{\vec{\rho}}(\theta_1)\right)\times\nabla_{R\perp}\phi\left(\vec{R}+\rho\hat{\vec{\rho}}(\theta_2)\right)-\frac{1}{2}\nabla_{R\perp}\bar{\phi}\left(\vec{R}\right)\times\nabla_{R\perp}\bar{\phi}\left(\vec{R}\right)$$

This can be calculated sampling the components of $\nabla\phi$ over a double gyro orbit.
e.g., 4-point θ_1 gyro orbit -> 10-point θ_1 - θ_2 double gyro orbit
8-point θ_1 gyro orbit -> 36-point θ_1 - θ_2 double gyro orbit

Again, at any given time, this is a function of the 4D $(\boldsymbol{R},\mu)$ phase space.



GYROKINETIC POISSON EQUATION

CASE 1: NEGLECT δf IN OPERATOR

Optimal first-order Pade approximant representation of Poisson operator

- can give factor of 3 accuracy improvement (~7% to ~2.5% relative error) for $0 \le b = k_\perp^2 \rho^2 \le 1$ over $\Gamma_0(b) \to 1/(1+b)$ replacement
- easily generalizable to multiple ion species
- no additional computational cost.

Multispecies Poisson/quasineutrality equation write $k_\perp^2 \rho_b^2 = x$, and $\rho_\alpha^2 / \rho_b^2 = \xi_\alpha$

$$L(x)\phi = \sum_\alpha R_\alpha \left[1 - \Gamma_0(\xi_\alpha x)\right]\phi = S$$

where

$$R_\alpha \equiv f_\alpha Z_\alpha T_e / T_\alpha$$

$$\rho_b^2 \equiv (m_b c / Z_b e B)^2 T_b / m_b$$
$$\xi_\alpha = \rho_\alpha^2 / \rho_b^2 = (m_\alpha / m_b)(T_\alpha / T_b)(Z_b / Z_\alpha)^2$$

Think of $k_\perp$ as the eigenvalue of $\nabla_\perp$

First-order Pade approximations are convenient

- inversion of either a Helmholtz or Poisson equation
- Eigenvalue of inverse operator is positive, decreasing with $k_\perp$, and bounded as $k_\perp \to \infty$.

The first-order Pade approximation to $L^{-1}(x)$:

$$L^{-1}(x) \approx \mathcal{L}(x) \equiv A + \frac{B}{C + x}.$$

A, B, C chosen to optimize the approximation.

Matching the asymptotic behavior of $L^{-1}(x)$.e., as $x \to 0$ gives

$$B = \lim_{x \to 0} x L^{-1}(x) = \left(\sum_{\alpha} R_{\alpha} x_{\alpha} \right)^{-1} \quad \text{and} \quad C = 0. \qquad (1.9)$$

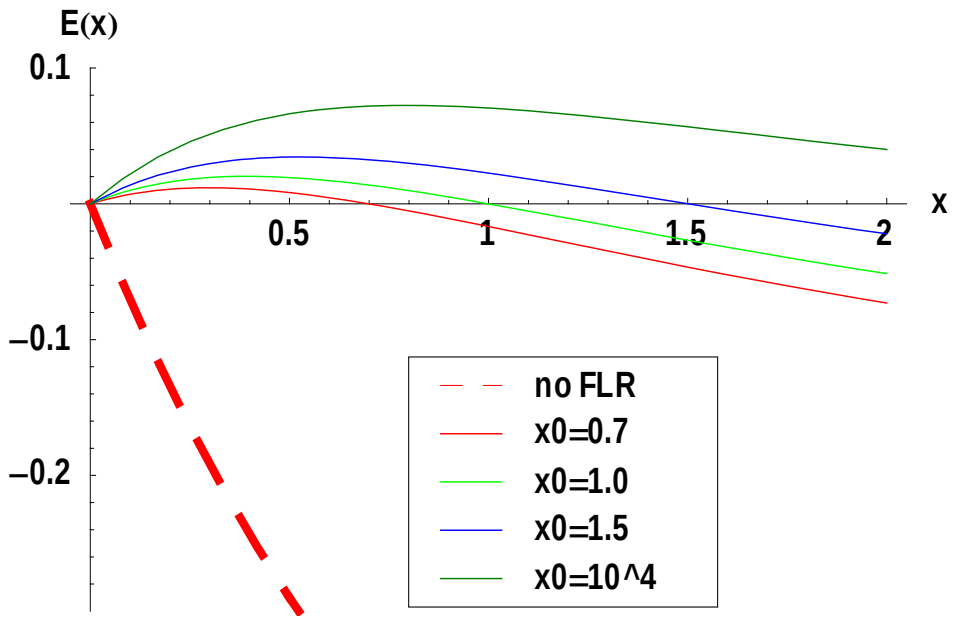
Matching $x = x_0$

$$A = \left(\sum_{\alpha} R_{\alpha} \left[1 - \Gamma_0(\xi_{\alpha} x_0) \right] \right)^{-1} - B / x_0 , \qquad (1.10)$$

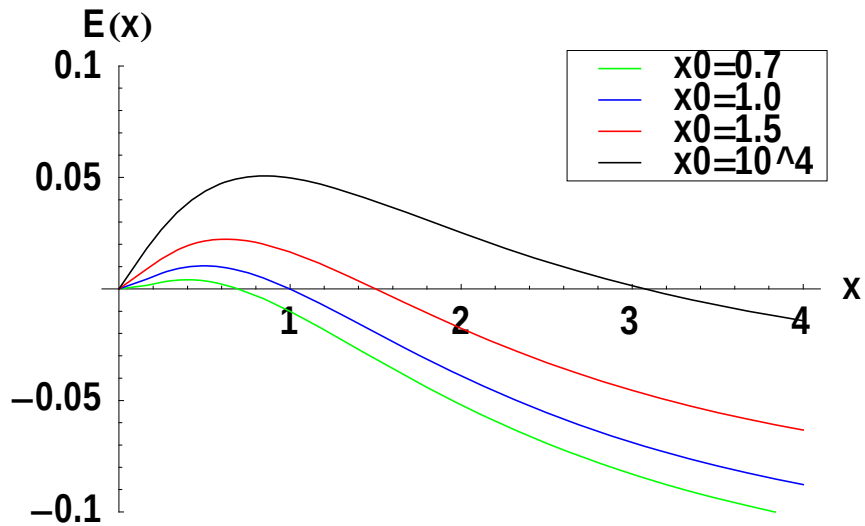
and the solution to the Pade-approximated version of Eq.(1.5) can be written formally as

$$\phi = \left[A + \frac{B}{x} \right] S . \qquad (1.11)$$

$\rightarrow$ solution of a single Poisson equation (and the addition of A times the source field).

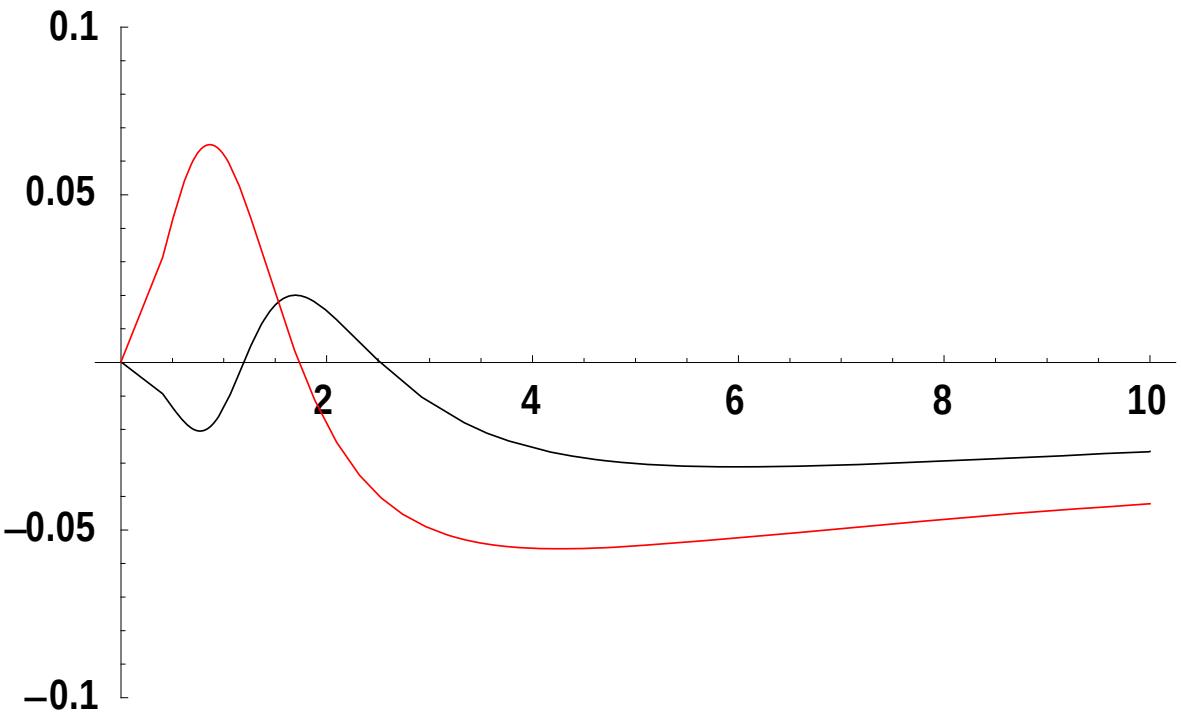


Plots of $E(x) \equiv \mathcal{L}(x)L(x) - 1$ x_0=*0.7*, *1.0*, *1.5*, and 10^4 (going from lower to upper curve), for single-species case



Plots of $E(x) \equiv \mathcal{L}(x)L(x) - 1$ x_0=*0.7, 1.0, 1.5*, and 10^4 Bulk deuterium + hot deuterium + carbon impurities; $(f_b,f_h,f_c)=(0.8,0.1,0.1)$, $(T_b,T_h,T_c)/\ T_b=(1,5,1)$, $(Z_b,Z_h,Z_c)=(1,1,6)$, $(A_b,A_h,A_c)=(2,2,12)$ $T_e=T_b$, $\Longrightarrow$ $(R_b,R_h,R_c)=(0.8,0.02,0.6)$, $(\xi_b, \xi_h, \xi_c)=(1,5,1)$; $B=1.5$ and A from x_0

Also, these can be optimized give reasonable (<7%) relative error for $0 \le b < \infty$. Improvement to <3% relative error can be obtained at some additional expense with an additional second-order Pade operator, which corresponds to the solution of a fourth-order elliptic PDE.



Plots of relative error for Pade approximations to the gyrokinetic Poisson operator optimized over a wide range of x for a single first-order Pade approximation (red curve) and for an optimized combination of a first-order operator followed by a second-order operator.

GYROKINETIC POISSON EQUATION
CASE 2: INCLUDE EVOLVING δf IN OPERATOR

ENERGY CONSERVATION CONSTRAINS THE OPTIONS FOR THE EQUATIONS OF MOTION AND POISSON EQUATIONS

Options:

1) Keep (a) full-F dependence or (b) an approximation to the operator with full-F dependence, e.g., long-wavelength approximation; Pade' approximation.

Energy conservation dictates a specific form for the Hamiltonian, given the particular gyrokinetic Poisson equation operator. Terms up to second-order in ε must be kept. The Hamiltonian and energy conservation relations are different in cases (a) and (b).

2) Keep terms in Hamiltonian only up to first order. Then energy conservation requires a fixed-ion-background approximation/model for the gyrokinetic Poisson operator. There may be various possibilities for the ion background: (a) uniform Maxwellian, (b) Maxwellian with a fixed nonuniform density (only), (c) Maxwellian with fixed nonuniform density and temperature profiles.

CONSTRUCTIVE PROCEDURE TO DETERMINE GYROCENTER-MOTION HAMILTONIAN, GIVEN THE GYROKINETIC POISSON OPERATOR

If the gyrokinetic Vlasov and field equations can be derived from an appropriate variational principle, then this system conserves energy.

$$\mathcal{A} = -\int d^8\mathcal{Z}\,\mathcal{F}_i(\mathcal{Z})\mathcal{H}_i(\mathcal{Z}) - \int d^8\mathcal{Z}\,\mathcal{F}_e(\mathcal{Z})\mathcal{H}_e(\mathcal{Z}) - \int \frac{d^4x}{4\pi}\left|\delta B\right|^2 ,$$

$$\frac{\delta\mathcal{A}}{\delta\mathcal{F}_i(\mathcal{Z})} = 0; \quad \delta\mathcal{F}_i = \left\{S, \mathcal{F}_i\right\} \quad \rightarrow \quad \text{ion GK Vlasov equation} \quad \left\{F_i, H_i\right\} = 0$$

$$\frac{\delta\mathcal{A}}{\delta\mathcal{F}_e(\mathcal{Z})} = 0; \quad \delta\mathcal{F}_i = \left\{S, \mathcal{F}_i\right\} \quad \rightarrow \quad \text{electron DK Vlasov equation} \quad \left\{F_e, H_e\right\} = 0$$

$$\frac{\delta\mathcal{A}}{\delta\phi(\boldsymbol{x})} = 0 \quad \rightarrow \quad \text{Poisson equation}$$

$$\frac{\delta\mathcal{A}}{\delta A_{1\|}(\boldsymbol{x})} = 0 \quad \rightarrow \quad \text{Ampere's law}$$

Conserved energy

$$\mathcal{E}_{gy} = \int d^3p\, F_i (H_i - q\left\langle T_{gy}^{-1}\phi_{gc}\right\rangle) - \int d^3p\, F_e (H_e + e\phi) + \frac{1}{8\pi}\left|\delta B\right|^2$$

Procedure – given Poisson equation (with operator Hermitian and linear in ϕ, construct an effective action $\mathcal{A}$ such that

$$\frac{\delta \mathcal{A}}{\delta\phi(\boldsymbol{x})} = 0 \quad \rightarrow \quad \text{Poisson equation}$$

$$L[F]\phi = S[F]$$

$$\mathcal{A} = \int dt\, dx\, \phi(x,t)\left[S[F](x,t) - \tfrac{1}{2}L[F]\phi(x,t)\right]$$

Identify this with

$$\mathcal{A} = -\int d^8\mathcal{Z}\, \mathcal{F}_i(\mathcal{Z})\mathcal{H}_i(\mathcal{Z})$$

Then $\mathcal{H}_i(\mathcal{Z}) = -\dfrac{\delta \mathcal{A}}{\delta \mathcal{F}_i(\mathcal{Z})}$ identifies ϕ-dependent part of $H_i(Z,t) = \mathcal{H}_i(\mathcal{Z}) + w$.

This identification is unique (up to an additive constant) because the Poisson tensor is non singular.

EXAMPLES – LONG-WAVELENGTH AND PADE APPROXIMATIONS

Gyrokinetic Poisson Equation:

$$0 \approx -\frac{1}{4\pi e}\nabla^2\phi(x) = \int d\Lambda\,\delta\left(\boldsymbol{R}+\boldsymbol{\rho}-\boldsymbol{x}\right)\left[F_i + \tilde{\phi}_1\frac{\partial F_i}{\partial \mu} + \boldsymbol{\nabla}_{R\perp}\tilde{\Phi}_1\cdot\boldsymbol{b_0}\times\boldsymbol{\nabla}_{R\perp}F_i\right] - n_e$$

Pade approximation – generalization of $\Gamma_0(x) \to \frac{1}{1+x}$ procedure to inhomogeneous F_i

$$0 \approx \boldsymbol{\nabla}_\perp\cdot\left[n_i\boldsymbol{\nabla}_\perp\left(\phi + \frac{\mu_{\text{th}}}{n_i}S\right)\right] + S = \boldsymbol{\nabla}_\perp\cdot\left[n_i\boldsymbol{\nabla}_\perp\left(\phi + \text{Pade term}\right)\right] + S$$

where

$$S = n_i - \frac{n_e}{Z} + \frac{1}{2}\boldsymbol{\nabla}_\perp^2 p_{\perp i}$$

Variational form

$$\mathcal{A} = -\int d\boldsymbol{x}\, n_i \left[\phi - \left(\nabla_\perp \phi\right)^2 / 2 \right] - \frac{1}{2}\int d\boldsymbol{x}\, p_{\perp i} \left(1 - \frac{\mu_{\text{th}}}{n_i} \nabla_\perp^2 \right) \nabla_\perp^2 \phi + \int d\boldsymbol{x}\, n_i \left(1 - \frac{n_e}{n_i} \right) \mu_{\text{th}} \nabla_\perp^2 \phi + \int d\boldsymbol{x}\, n_e \phi$$

$$= -\int d\boldsymbol{x}\, n_i \left[\phi - \left(\nabla_\perp \phi\right)^2 / 2 \right] - \frac{1}{2}\int d\boldsymbol{x}\, p_{\perp i} \nabla_\perp^2 \phi - n_e + \text{Pade terms}$$

Ion Hamiltonian

$$H = \phi - \frac{1}{2}\left(\nabla_\perp \phi\right)^2 + \frac{1}{4}\rho_i^2 \nabla_\perp^2 \phi + \text{Pade terms}$$

Conclusion: Gyrokinetic ion equations of motion are incompatible with the long-wavelength approximation to the GK Poisson (quasineutrality; vorticity) equation.

FINITE-ELEMENT DISCRETIZATION OF THE GYROKINETIC POISSON EQUATION

A straightforward discretization scheme for the full-*F* gyrokinetic Poisson equation results from a Finite-element representation, i.e., weak form + Galerkin representation of ϕ. All derivatives and gyroaveraging operations can be recast into derivatives operating on the Galerkin basis functions.

Begin with gyrokinetic Poisson equation:

Gyrokinetic Poisson Equation:

$$L\phi = S$$

$$L\phi = \frac{1}{4\pi}\nabla^2\phi(x) + \int d\Lambda\,\delta\left(\boldsymbol{R}+\boldsymbol{\rho}-\boldsymbol{x}\right)\left[\tilde{\phi}_1\frac{\partial F_i}{\partial\mu} + \boldsymbol{\nabla}_{R\perp}\tilde{\Phi}_1\cdot\boldsymbol{b_0}\times\boldsymbol{\nabla}_{R\perp}F_i\right]$$

$$S = n_e - \int d\Lambda\,\delta\left(\boldsymbol{R}+\boldsymbol{\rho}-\boldsymbol{x}\right)F_i\left(\boldsymbol{R}\right);\; d\Lambda = d\boldsymbol{R}\,dv_{\|}\,B\,d\mu\,d\theta$$

Procedure: use Galerkin discretization of

$$\frac{\delta\mathcal{A}}{\delta\phi(\boldsymbol{x})} = 0$$

$$0 = \int d\boldsymbol{x}\phi\left(S - \tfrac{1}{2}L\phi\right)$$

$$= \int d\boldsymbol{x}\phi S + \frac{1}{8\pi}\int d\boldsymbol{x}\left(\nabla\phi\right)^2 + \frac{1}{2}\int d\Lambda F_i \frac{\partial\tilde{\phi}^2}{\partial\mu}$$

$$-\frac{1}{2}\int d\Lambda F_i\left[\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}_{R\perp}\tilde{\phi}\times\boldsymbol{\nabla}_{R\perp}\tilde{\Phi} - \frac{1}{B}\tilde{\phi}\boldsymbol{\nabla}_{R\perp}\tilde{\Phi}\cdot\boldsymbol{J}_{\text{eqm}}\right]$$

$$\phi(\boldsymbol{x}) = \sum_{\boldsymbol{l}}\phi_{\boldsymbol{l}}\,\psi_{\boldsymbol{l}}(\boldsymbol{x});\ \boldsymbol{l} = (i,j)$$

Insert Galerkin representation

$$\psi_{i,j} = \psi\left(\frac{x - x_{i,j}}{\Delta_{x;i,j}}, \frac{y - y_{i,j}}{\Delta_{y;i,j}}\right)$$

$\rightarrow$ resolves derivatives onto basis functions

$$\frac{\delta\left(\text{weak form Galerkin GK Poisson}\right)}{\delta\phi_{i,j}} = 0 \rightarrow$$ Matrix equation to be solved

$$\boldsymbol{M}^{\mathrm{S}} \cdot \boldsymbol{\varphi} = \boldsymbol{S}$$

$$\boldsymbol{M}^{\mathrm{S}} = \left(\boldsymbol{M} + \boldsymbol{M}^{\mathrm{T}}\right)/2$$

$$M_{\boldsymbol{k},\boldsymbol{l}} = \frac{1}{4\pi}\int d\boldsymbol{x} \nabla \psi_{\boldsymbol{k}} \cdot \nabla \psi_{\boldsymbol{l}} + \int d\Lambda F_i \frac{\partial\left(\tilde{\psi}_{\boldsymbol{k}}\tilde{\psi}_{\boldsymbol{l}}\right)}{\partial \mu}$$

$$- \int d\Lambda F_i \left[\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}_{R\perp}\tilde{\psi}_{\boldsymbol{k}} \times \boldsymbol{\nabla}_{R\perp}\tilde{\Psi}_{\boldsymbol{l}} - \frac{1}{B}\tilde{\psi}_{\boldsymbol{k}}\boldsymbol{\nabla}_{R\perp}\tilde{\Psi}_{\boldsymbol{l}} \cdot \boldsymbol{J}_{\mathrm{eqm}}\right]$$

$$S_{\boldsymbol{k}} = \int d\boldsymbol{x}\, S\, \psi_{\boldsymbol{k}}$$

“Deposition” or projection from Z to x is needed to calculate the matrix elements.

The resulting matrices are still sparse - for solution on perpendicular slices with N^2 cells of size $\sim\rho_i$, about 100 N^2 nonzero matrix elements are needed (not N^4).

CONCLUSIONS

Gyrokinetic-based models for edge plasmas have been developed, and key issues in their implementation have been solved.

System: Ions - electromagnetic gyrokinetic - a practical simplification of the electromagnetic ion terms, consistent with energy conservation, has been found; Electrons - drift kinetic

Can neglect higher-order magnetic inhomogeneity terms

Key consequence of energy conservation for edge gyrokinetic models: Working to second order in GK parameter (i.e., keeping time-dependent δf in GK Poisson operator and matching second-order terms in equations of motion) is challenging.

– energy conservation -> must match equations of motion to field solve

Implementation issues:

Have shown how to discretize – 2nd-order non-δf GK Poisson equation
– 2nd-order equations of motion

Improved accurate and easily discretized Pade' approximations to GK Poisson operator have been found. Questions about energy conservation for these (and long-wavelength approximation) need to be addressed before attempting to use them.

Examined theory for gyrokinetic Coulomb collision operator. A correct model is similar to that formulated by Brizard, except that first order terms need to be kept in the pull-back transformation in the transformation from the gyrocenter distribution function to the "physical-space" distribution function before this is inserted into the Landau-Fokker-Planck operator.

Ongoing work

Develop a practical/optimal scheme for implementing a bilinear gyrokinetic collision operator.

Develop a practical and correct way to calculate the profile-scale zonal potentials within a gyrokinetic treatment (c.f. recent work by Para and Catto).